

## Fractional M2-branes

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**ABSTRACT:** We consider two generalizations of the  $\mathcal{N} = 6$  superconformal Chern-Simons-matter theories with gauge group  $U(N) \times U(N)$ . The first generalization is to  $\mathcal{N} = 6$  superconformal  $U(M) \times U(N)$  theories, and the second to  $\mathcal{N} = 5$  superconformal  $O(2M) \times USp(2N)$  and  $O(2M + 1) \times USp(2N)$  theories. These theories are conjectured to describe M2-branes probing  $C^4/\mathbf{Z}_k$  in the unitary case, and  $C^4/\widehat{\mathbf{D}}_k$  in the orthogonal/symplectic case, together with a discrete flux, which can be interpreted as  $|M - N|$  fractional M2-branes localized at the orbifold singularity. The classical theories with these gauge groups have been constructed before; in this paper we focus on some quantum aspects of these theories, and on a detailed description of their M theory and type IIA string theory duals.

**KEYWORDS:** AdS-CFT Correspondence, Duality in Gauge Field Theories, Extended Supersymmetry, Chern-Simons Theories.

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**1. Introduction and summary of results**

Motivated by the work of Bagger and Lambert [1] (see also [2, 3]–[4]), there has recently been great interest in studying three dimensional superconformal field theories and their relation to the low-energy theory on M2-branes. The attempts to directly generalize the Bagger-Lambert construction of an  $\mathcal{N} = 8$  superconformal theory with  $SO(4)$  gauge group to the low-energy theory on  $N$  M2-branes have not yet been successful, and seem to lead either to non-unitary theories or to non-conformal theories. In [5] we suggested an alternative route to studying M2-branes. We presented a  $U(N)_k \times U(N)_{-k}$  Chern-Simons matter theory with explicit  $\mathcal{N} = 6$  superconformal symmetry.<sup>1</sup> Considering the brane construction of this theory led us to conjecture that it was equivalent to the low-energy theory of  $N$  M2-branes on a  $\mathbf{C}^4/\mathbf{Z}_k$  singularity, and we presented various pieces of evidence for this conjecture. In the special cases of  $k = 1$  and  $k = 2$  this conjecture implies that the theories should have an enhanced  $\mathcal{N} = 8$  supersymmetry, whose generators cannot be described

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<sup>1</sup>These theories are special cases of  $\mathcal{N} = 4$  superconformal Chern-Simons theories constructed in [6, 7].

locally in terms of the original fields of the theory. In particular, for  $k = 1$  we conjectured that the  $U(N)_1 \times U(N)_{-1}$  theory is equivalent to the low-energy theory on  $N$  M2-branes in flat space.

In this paper we generalize the construction of [5] in two directions, both of which break the parity symmetry of the original construction. Both of these generalizations have already been considered as classical field theories in [8], but we add more details about quantum aspects of these theories, and we also discuss in detail their gravitational duals. The first generalization we consider is to  $U(M)_k \times U(N)_{-k}$  Chern-Simons-matter theories, with the same matter content and interactions as in [5], but with  $M \neq N$ . From the point of view of M2-branes probing a  $\mathbf{C}^4/\mathbf{Z}_k$  singularity, these theories arise (for  $M > N$ ) when we have  $(M - N)$  fractional M2-branes sitting at the singularity, together with  $N$  M2-branes that are free to move around. On the field theory side, classically this generalization is straightforward, and still leads to  $\mathcal{N} = 6$  superconformal field theories. However, we will argue (both directly in the field theory, and by using a brane construction) that quantum mechanically these theories only exist as unitary superconformal field theories when  $|M - N| \leq |k|$ . For these cases, we argue that the gravitational dual is the same  $AdS_4 \times S^7/\mathbf{Z}_k$  background described in [5], just with an additional “torsion flux”, which takes values in  $H^4(S^7/\mathbf{Z}_k, \mathbf{Z}) = \mathbf{Z}_k$ . This flux may be thought of as an M theory analog of orbifold discrete torsion, corresponding to a discrete holonomy of the M theory 3-form field on a torsion 3-cycle in  $S^7/\mathbf{Z}_k$ . This is a good description of the gravitational dual when  $N \gg k^5$ . For  $k \ll N \ll k^5$ , on the other hand, the appropriate description is in terms of type IIA string theory on  $AdS_4 \times \mathbf{CP}^3$ , with a discrete holonomy of the NSNS 2-form field on the  $\mathbf{CP}^1 \subset \mathbf{CP}^3$ . In this description the NSNS 2-form holonomy is quantized dynamically.

Our second generalization involves an orientifold in the original brane construction, which changes the gauge symmetry to  $O(M) \times USp(2N)$ , and reduces the charged matter content to two bi-fundamental chiral multiplets. The resulting gauge theories have  $\mathcal{N} = 5$  superconformal symmetry, and again there are bounds on  $|M - 2N|$  in order for the quantum theory to exist (which we will describe in more detail below). We conjecture that these theories are dual to the low-energy theory on M2-branes probing a  $\mathbf{C}^4/\widehat{\mathbf{D}}_k$  singularity, where  $\widehat{\mathbf{D}}_k$  is the binary dihedral group with  $4k$  elements, and (generically) with some discrete flux, taking values in  $H^4(S^7/\widehat{\mathbf{D}}_k, \mathbf{Z}) = \mathbf{Z}_{4k}$ , at the singularity. As before, one can also think of this discrete flux as associated with fractional M2-branes sitting at the singularity. The gravitational dual of these theories may thus be described (for  $N \gg k^5$ ) by M theory on  $AdS_4 \times S^7/\widehat{\mathbf{D}}_k$ , possibly with discrete torsion. When  $k \ll N \ll k^5$  this background reduces to an orientifold of type IIA string theory on  $AdS_4 \times \mathbf{CP}^3$ , again with a discrete holonomy of the NSNS 2-form.

Our study provides additional examples of  $AdS_4/CFT_3$  duals with extended supersymmetry, which should hopefully be useful for understanding better the properties of this duality. In particular, we extend the duality to theories which are not parity-invariant, and are thus closer in spirit to the Chern-Simons-matter theories encountered in condensed matter applications. It would be interesting to analyze these theories in the 't Hooft limit (large  $N$  and  $k$  with fixed  $N/k$ ) and to see if integrability (on the field theory or string theory sides of the duality [9, 10]) may be used to learn about various properties of these

theories. Some of our results also have applications to more general three dimensional theories. For instance, our claim (based on [11]) that the  $U(M)_k \mathcal{N} = 3$  supersymmetric pure Chern-Simons theories with  $M > k$  do not exist implies that almost all of the classical supersymmetric vacua of the mass-deformed  $U(N) \times U(N)$  theory found in [12] do not survive in the quantum theory (when  $k = 1$ ), so more vacua need to be found in order to match with expectations.<sup>2</sup>

The rest of the paper is organized as follows. In section 2 we consider the  $\mathcal{N} = 6$   $U(M) \times U(N)$  superconformal Chern-Simons-matter theories, their realization in terms of brane configurations in type IIB string theory, and their gravity dual descriptions. In section 3 we consider the  $\mathcal{N} = 5$   $O(M) \times USp(2N)$  superconformal theories, their type IIB brane realizations, and their gravity dual descriptions. The moduli space of these theories is analyzed in an appendix.

## 2. $U(M) \times U(N)$ theories

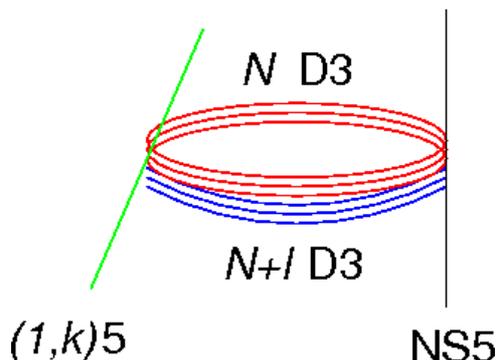
It is straightforward to generalize the field theory construction of [5] to supersymmetric  $U(M)_k \times U(N)_{-k}$  Chern-Simons matter theories. As in the case of  $N = M$ , we can take the  $\mathcal{N} = 3$  supersymmetric Chern-Simons-matter theory with this gauge group and with two bi-fundamental hypermultiplets, and argue that because of the special form of its superpotential, it in fact has an enhanced  $\mathcal{N} = 6$  supersymmetry (these theories were also recently discussed in [8]). Many aspects of the analysis of these theories are very similar to the case of  $N = M$  which was discussed in [5]. For instance, the moduli space of these theories turns out to be exactly the same as that of the  $U(L)_k \times U(L)_{-k}$  theories with  $L = \min(N, M)$ , and the spectrum of non-baryonic chiral primary operators is also the same as the one in that theory. The main difference is that the effective field theory on the moduli space contains an extra  $U(M - N)_k$  Chern-Simons (CS) theory (when  $M > N$ ), with no massless charged fields. In this section we will describe the gravitational dual of the  $U(M) \times U(N)$  theories, and some of the differences between these theories and the  $N = M$  theories.

### 2.1 The field theories

Let us assume to begin with that  $M > N$ , and consider the superconformal theories  $U(N + l)_k \times U(N)_{-k}$  with  $l \geq 0$  (and  $k > 0$ ). Parity takes this theory to the  $U(N)_k \times U(N + l)_{-k}$  theory. We will see later that there is also an equivalence relating a theory of the first type to a theory of the second type (with  $l \rightarrow k - l$ ). As in the equal rank case, the more general superconformal theory can be obtained (at least classically) as the IR limit of an  $\mathcal{N} = 3$  supersymmetric gauge theory, arising as the low-energy limit of a type IIB brane construction. Recall that the brane construction of the  $U(N)_k \times U(N)_{-k}$  theories involves [5]  $N$  D3-branes winding around a circle, and intersecting an NS5-brane and a  $(1, k)5$ -brane (a bound state of an NS5-brane with  $k$  D5-branes) at specific angles (described in [5]). To get the  $U(N + l)_k \times U(N)_{-k}$  theory we add to this  $l$  D3-branes which

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<sup>2</sup>We thank M. Van Raamsdonk for discussions on this issue.



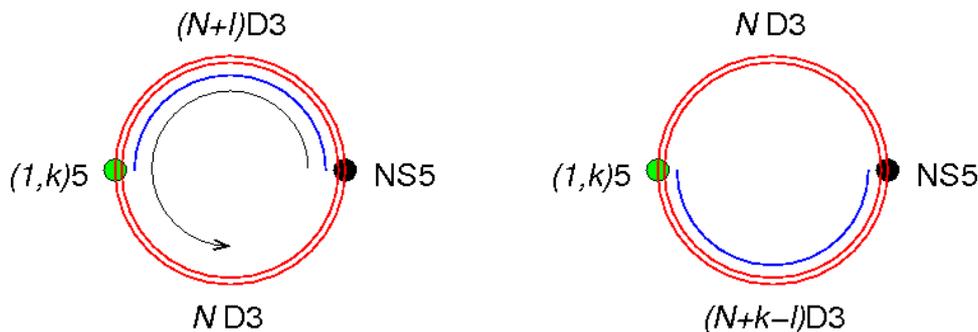
**Figure 1:** The brane construction of the  $U(N+l)_k \times U(N)_{-k}$  theory.

are suspended between the NS5-brane and the  $(1, k)5$ -brane on one side of the circle (the side in which the 5-brane intersection numbers lead to a Chern-Simons level of  $+k$ , see figure 1). This construction makes it clear that the classical moduli space is identical to the moduli space of the  $U(N)_k \times U(N)_{-k}$  theory. It corresponds to the motion of the  $N$  free D3-branes (as well as to the Wilson line and dual gauge field on the D3-branes). The  $l$  suspended D3-branes are locked into position by the two 5-branes, so there is no moduli space associated with them.

There is some additional “braneology” associated with these configurations, which is reflected in interesting properties of the corresponding field theories.<sup>3</sup> First, let us try to go onto the classical moduli space, by separating the  $N$  branes which wrap all the way around the circle from the other branes. This leaves only  $l$  “fractional branes” stretched between the two five-branes. For  $l > k$ , it was argued in [11] that this configuration of “fractional branes” breaks supersymmetry. In the brane picture this follows from the “s-rule” [13] forbidding more than one D3-brane from ending on a specific NS5-brane D5-brane pair, which was generalized to this case in [11]. From the field theory point of view, at such a point on the moduli space the low-energy field theory includes a pure  $\mathcal{N} = 3$   $U(l)_k$  YM-CS theory living on the fractional D3-branes. A simple generalization of the Witten index computation of [14] implies that supersymmetry is unbroken for  $l \leq k$ . However, for  $l > k$ , this computation suggests that supersymmetry is broken [11]. This implies that the classical moduli space of this  $U(N+l)_k \times U(N)_{-k}$  brane configuration is at least partly lifted when  $l > k$ .

The second interesting bit of brane physics has to do with the brane creation effect [13]. When the NS5-brane crosses the  $(1, k)5$ -brane,  $k$  D3-branes are created between them, corresponding to one D3-brane for each D5-brane component of the  $(1, k)5$ -brane. If there are  $l$  suspended D3-branes to begin with, there will be  $(k - l)$  suspended D3-branes after the crossing. Let us start with the brane configuration for the  $\mathcal{N} = 3$  gauge theory with  $U(N+l)_k \times U(N)_{-k}$  where  $0 \leq l \leq k$ , and move one of the 5-branes around the circle in the direction that initially shortens the suspended 3-branes. The final configuration then describes the  $\mathcal{N} = 3$  gauge theory with  $U(N)_k \times U(N+k-l)_{-k}$ , see figure 2. We thus

<sup>3</sup>We thank A. Hashimoto for useful discussions on these brane configurations.



**Figure 2:** Branes are created when we move a 5-brane around the circle.

expect these two gauge theories to flow to the same superconformal theory. Below we will argue that these two superconformal theories are indeed equivalent.<sup>4</sup>

Let us now consider the  $\mathcal{N} = 6$  superconformal  $U(N+l)_k \times U(N)_{-k}$  Chern-Simons-matter theory that is obtained in the IR limit of the above gauge theory. We will use the description of these theories in [5], in which they are written as special cases of  $\mathcal{N} = 3$  supersymmetric Chern-Simons-matter theories that happen to have enhanced supersymmetry. We will now argue that the theories with  $l > k$  do not exist as unitary conformal field theories, and that the above theory with  $l \leq k$  is equivalent to the theory with  $U(N)_k \times U(N+k-l)_{-k}$ .

In Chern-Simons-matter theories with this much supersymmetry the metric on the moduli space cannot be corrected, so we can go to a generic point on the moduli space, where at low energy we get  $N$  copies of a  $U(1)_k \times U(1)_{-k}$  theory (as in the  $U(N)_k \times U(N)_{-k}$  case), together with a pure  $U(l)_k$   $\mathcal{N} = 3$  supersymmetric Chern-Simons theory, with no massless charged matter fields. Naively, this supersymmetric Chern-Simons theory is the same as the bosonic Chern-Simons theory at level  $k$ , since the other fields in the  $\mathcal{N} = 3$  vector multiplet just have Gaussian actions in the absence of any matter fields, but in fact we should be more careful. Recall that the Chern-Simons level in [5] was defined such that it was equal to the level of the  $\mathcal{N} = 3$  Yang-Mills-Chern-Simons theory which flows to the Chern-Simons-matter theory in the IR. In this theory the level is not renormalized [15], but this comes from the cancelation of two contributions. There is a shift of  $k \rightarrow k - l \cdot \text{sign}(k)$  coming from integrating out the gauginos, and another opposite shift from the one-loop contributions of the gauge field itself (these shifts are only in the  $SU(l)$  level, not in the  $U(1)$  level, but this will not affect our arguments). Thus, if we think of this theory in bosonic terms, we can say that it includes a bosonic  $SU(l)_{k-l \cdot \text{sign}(k)}$  Chern-Simons theory. However this only makes sense if  $|k| \geq l$ , since otherwise the one-loop shift in the bosonic Chern-Simons theory (which generally takes  $k \rightarrow k + l \cdot \text{sign}(k)$ ) would have the wrong sign and would not bring us back to the level  $k$  theory that we want. This suggests that the  $\mathcal{N} = 3$  supersymmetric  $U(l)_k$  pure Chern-Simons theory does not exist as a unitary theory

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<sup>4</sup>Naively, moving the 5-brane around the circle multiple times, or in the other direction, should produce more equivalences. However, all of these additional configurations lead either to anti-D3-branes or to  $U(N'+l')_k \times U(N')_{-k}$  theories with  $l' > k$ .

for  $l > k$  (at least, it does not seem to be equivalent to any standard bosonic Chern-Simons theory). We are therefore led to conjecture that the superconformal  $U(N+l)_k \times U(N)_{-k}$  theories with  $l > k$  do not exist as unitary theories. Note that these theories are never weakly coupled (recall that a  $U(N)_k$  theory is weakly coupled only when  $|k| \gg N$ ). This conjecture is consistent with what we found in the corresponding  $\mathcal{N} = 3$   $U(N+l)_k \times U(N)_{-k}$  Yang-Mills-Chern-Simons theories, where we showed that for  $l > k$  the moduli space was partly lifted, so these theories cannot flow to the  $U(N+l) \times U(N)$  Chern-Simons-matter theories (whose moduli space does not receive quantum corrections).

For the superconformal theories with  $l \leq k$  the brane picture suggested that there should be an equivalence between pairs of theories of the form

$$U(N+l)_k \times U(N)_{-k} = U(N)_k \times U(N+k-l)_{-k}. \tag{2.1}$$

Note that again at least one of these two theories is always strongly coupled, so it is not easy to verify this duality<sup>5</sup>. As (weak) evidence for this conjectured equivalence, note that on the moduli space one obtains the  $U(l)_k$  supersymmetric CS theory for the theory on the left-hand side of (2.1), and the  $U(k-l)_{-k}$  theory for the theory on the right-hand side. The corresponding effective bosonic theories therefore include pure  $SU(l)_{k-l}$  CS and pure  $SU(k-l)_{-l}$  CS, respectively. In the presence of a boundary, we obtain the corresponding WZW theories,  $SU(l)_{k-l}$  and  $SU(k-l)_l$ , which are precisely related by level-rank duality. For the special case  $l = k$  we get an equivalence between the  $U(N+k)_k \times U(N)_{-k}$  theory and the original  $l = 0$  theory with  $U(N)_k \times U(N)_{-k}$ . Note that in this case on the moduli space of the former theory we seem to obtain an extra  $U(k)_k$  pure Chern-Simons theory, but as argued above, at least the  $SU(k)$  part of this is equivalent to a bosonic  $SU(k)_0$  theory, which is an empty theory.<sup>6</sup> We therefore end up with a total of  $k$  inequivalent theories for any given minimal rank  $N$  (which sets the dimension of the moduli space to be  $8N$ ). Below we will see that this is consistent with the dual gravitational picture. Note that, as expected, for the case  $k = 1$  corresponding to M2-branes in flat space we do not find any new theories associated with “fractional branes”.

## 2.2 M theory lift

Lifting the type IIB brane configuration for the  $U(N+l)_k \times U(N)_{-k}$  theory to M theory as in [5], we obtain  $N$  M2-branes moving in  $\mathbf{C}^4/\mathbf{Z}_k$ , together with  $l$  “fractional M2-branes”, which correspond to M5-branes wrapped on a vanishing 3-cycle at the orbifold point. This is a pure torsion cycle since  $H_3(S^7/\mathbf{Z}_k, \mathbf{Z}) = \mathbf{Z}_k$ . In other words,  $k$  wrapped M5-branes are equivalent to none, and there are  $k$  inequivalent configurations. This can be thought of as an M theory analog of orbifold discrete torsion [17], classified by  $H^3(\Gamma, U(1))$ ,

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<sup>5</sup>In particular, these theories seem to have a different number of degrees of freedom, but this number can be significantly modified at strong coupling.

<sup>6</sup>There is also a remaining  $U(1)_k$  pure CS theory. We will see below that in their dual description these two theories differ by a shift of a bulk theta angle by  $2\pi$ . The extra  $U(1)_k$  part may be related to the fact that the boundary theory changes by a  $U(1)$  Chern-Simons theory when the theta angle in the bulk is shifted by  $2\pi$  [16].

where  $\Gamma = \mathbf{Z}_k$ . This discrete torsion may be realized by a discrete holonomy of the 3-form potential on the above 3-cycle,  $\exp(i \int_{3\text{-cycle}} C_3) \in \mathbf{Z}_k$ . By Poincaré duality we can also associate the fractional branes to a pure torsion flux of the 4-form field strength in  $H^4(S^7/\mathbf{Z}_k, \mathbf{Z}) = \mathbf{Z}_k$ . We therefore conclude that the  $U(N+l)_k \times U(N)_{-k}$  superconformal theory with  $0 \leq l \leq k-1$  describes  $N$  M2-branes on  $\mathbf{C}^4/\mathbf{Z}_k$  with  $l$  units of discrete torsion. Our analysis of the discrete torsion generalizes the known result in the case of  $k=2$ , where the two variants of  $\mathbf{C}^4/\mathbf{Z}_2$  are the  $OM2^+$  and  $OM2^-$  orbifold planes [18, 19].

Note that the theory of two M2-branes on the  $\mathbf{C}^4/\mathbf{Z}_2$  orbifold with discrete torsion was also claimed [20, 21] to be described by the  $\mathcal{N} = 8$  Bagger-Lambert theory, with gauge group  $SU(2) \times SU(2)$ , at level  $k=2$ . Here we claim that the same theory may be described using the  $\mathcal{N} = 6$   $U(3)_2 \times U(2)_{-2}$  Chern-Simons-matter theory, which has a hidden  $\mathcal{N} = 8$  superconformal symmetry (as well as a hidden parity symmetry).

The number of discrete torsion variants is consistent with the restriction  $l \leq k$ , together with the equivalence of the  $l=0$  and  $l=k$  theories, found above for the superconformal field theories. In fact the more general equivalence between the pairs of theories in (2.1) can be understood as a parity symmetry in M theory, that reflects one of the coordinates along the M2-branes and at the same time takes  $C_3 \rightarrow -C_3$ . The former has the effect of changing the sign of the Chern-Simons term, and therefore the level  $k \rightarrow -k$ , and the latter has the effect of changing the torsion class  $[G_4] \rightarrow k - [G_4]$ , and therefore  $l \rightarrow k-l$  in the gauge group that has level  $(-k)$ .

### 2.3 M theory dual

Taking the near-horizon limit of the brane configuration described above as in [5], we find that our field theories are dual to M theory on  $AdS_4 \times S^7/\mathbf{Z}_k$ , with  $N$  units of 4-form flux in  $AdS_4$ , and  $l$  units of discrete torsion:

$$\begin{aligned} ds^2 &= \frac{R^2}{4} ds_{AdS_4}^2 + R^2 ds_{S^7/\mathbf{Z}_k}^2, \\ G_4 &\sim N \epsilon_4, \\ \int_{S^3/\mathbf{Z}_k \subset S^7/\mathbf{Z}_k} C_3 &= 2\pi \frac{l}{k}, \end{aligned} \tag{2.2}$$

where  $\epsilon_4$  is the unit volume form on  $AdS_4$  and  $R = (2^5 \pi^2 k N)^{1/6} l_p$ . For  $N \gg k^5$  the theory has a good approximation in terms of eleven dimensional supergravity on this background. We can describe  $S^7/\mathbf{Z}_k$  in terms of 4 complex coordinates  $z_i$  with a constraint  $\sum |z_i|^2 = 1$  and an identification  $z_i \sim e^{2\pi i/k} z_i$ . In this language, the 3-cycle  $S^3/\mathbf{Z}_k$  can be described (for instance) as  $\{|z_1|^2 + |z_2|^2 = 1, z_3 = z_4 = 0\}$ .

We can also describe the  $S^7/\mathbf{Z}_k$  as an  $S^1$  fibred over  $\mathbf{CP}^3$ , in which case its metric is expressed as

$$ds_{S^7/\mathbf{Z}_k}^2 = \frac{1}{k^2} (d\varphi + k\omega)^2 + ds_{CP^3}^2, \tag{2.3}$$

where  $\varphi \sim \varphi + 2\pi$ , and  $d\omega = J$  where  $J$  is the Kähler form on  $\mathbf{CP}^3$ . The 3-form which has the above holonomy can then be expressed locally as

$$C_3 \propto \frac{l}{k} J \wedge d\varphi. \tag{2.4}$$

## 2.4 Type IIA string theory dual

Recall that for  $M = N$ , when  $N^{1/5} \ll k \ll N$ , the M theory circle becomes small (in eleven dimensional Planck units). A better description of the background dual to the  $U(N)_k \times U(N)_{-k}$  theory is obtained [5] by reducing to type IIA string theory on  $AdS_4 \times \mathbf{CP}^3$ , with  $N$  units of  $\tilde{F}_4$  flux on  $AdS_4$  and  $k$  units of  $F_2$  flux on the  $\mathbf{CP}^1$  2-cycle in  $\mathbf{CP}^3$ ; we can write this background (setting the string scale to one) as

$$\begin{aligned} ds_{\text{string}}^2 &= \frac{R^3}{k} \left( \frac{1}{4} ds_{AdS_4}^2 + ds_{CP^3}^2 \right), \\ e^{2\phi} &= \frac{R^3}{k^3} \sim \frac{N^{1/2}}{k^{5/2}} = \frac{1}{N^2} \left( \frac{N}{k} \right)^{5/2}, \\ \tilde{F}_4 &= \frac{3}{8} R^3 \hat{e}_4, \\ F_2 &= k d\omega = kJ, \end{aligned} \tag{2.5}$$

where the radius of curvature in string units is  $R_{\text{str}}^2 = R^3/k = 2^{5/2} \pi \sqrt{N/k}$ .

To describe the  $U(N+l)_k \times U(N)_{-k}$  theories we need to add to this background the reduction of the discrete torsion in M theory. The 3-form in (2.4) has one leg on the M theory circle, so it reduces to the NSNS 2-form  $B_2$ , which attains a non-trivial holonomy on the  $\mathbf{CP}^1$  2-cycle in  $\mathbf{CP}^3$

$$b_2 \equiv \frac{1}{2\pi} \int_{\mathbf{CP}^1 \subset \mathbf{CP}^3} B_2 = \frac{l}{k}. \tag{2.6}$$

Note that we are using here units in which  $b_2$  is periodic with period one. Equation (2.6) seems like a surprising result at first, since the holonomy of  $B_2$  is a continuous variable, i.e. there is no discrete torsion in  $\mathbf{CP}^3$ . Let us therefore explain how it arises directly in the type IIA description of the fractional branes.

The M5-branes wrapped on  $S^3/\mathbf{Z}_k$  reduce to D4-branes wrapped on the  $\mathbf{CP}^1$  2-cycle of  $\mathbf{CP}^3$ . Note that the low-energy theory on  $l$  such D4-branes contains a  $U(l)$  Chern-Simons term at level  $k$ , as we expect, due to the  $F_2 \wedge A \wedge dA$  term in the D4-brane effective action (where  $A$  is the D4-brane worldvolume gauge field). From the point of view of the theory on  $AdS_4$ , the D4-branes are domain walls separating the space into two separate regions. Of course, these domain walls (which we think of as filling space and sitting at a fixed radial position) are not static, but rather they feel a force driving them towards the horizon of  $AdS_4$ ; this corresponds to the fact that there is no moduli space for moving the fractional branes around. Nevertheless, let us assume for a moment that we fix the branes at some fixed radial position. The D4-branes are a source for the RR 5-form field  $C_5$ ; this source means that as we go from one side of the D4-branes to the other, the flux of the RR 4-form field strength  $\tilde{F}_4$ , integrated over  $\mathbf{CP}^2 \subset \mathbf{CP}^3$  (which is the dual cycle to the one the branes are wrapped on), jumps by  $l$  units.

Naively this suggests that the type IIA dual of the  $U(N+l) \times U(N)$  theories should be described by a configuration with  $l$  units of 4-form flux on  $\mathbf{CP}^2 \subset \mathbf{CP}^3$ . However, such a solution does not seem to exist, and we wish to claim that the correct background is

actually the one described above. The point is that we also have  $k$  units of  $F_2$  flux on  $\mathbf{CP}^1$ , and in the presence of this flux the equation of motion of  $\tilde{F}_4$  is modified to  $d\tilde{F}_4 = -F_2 \wedge H_3$ .<sup>7</sup> From the point of view of the effective field theory on  $AdS_4$ , this means that  $f_4 \propto \int_{\mathbf{CP}^2} \tilde{F}_4$  (normalized so that it is quantized to be an integer) is not conserved, but rather its equation of motion is given by  $d(f_4 + kb_2) = 0$ . This means that the conserved flux which jumps by  $l$  units as we cross the D4-branes is not  $f_4$  but rather  $f_4 + kb_2$ , and another way to realize such a jump is by having  $b_2$  jump by  $l/k$ . This is precisely what we obtained above. Thus, we conjecture that the type IIA string theory dual of the  $U(N+l)_k \times U(N)_{-k}$  theory is the original  $AdS_4 \times \mathbf{CP}^3$  background, together with a  $B$  field holonomy  $b_2 = l/k$ . Clearly this is a solution of the classical equations of motion of type IIA supergravity (preserving the same supersymmetries as the original background), since  $b_2$  does not enter the equations.

Note that naively  $b_2$  is not quantized, but the arguments above suggest that  $(f_4 + kb_2)$  is quantized, implying that when  $f_4 = 0$ ,  $b_2$  must be quantized in units of  $1/k$ . The periodicity  $b_2 \sim b_2 + 1$  then implies that there are  $k$  inequivalent theories, in agreement with both M theory and the field theory arguments. As a consistency check, note that the equivalence of  $b_2 \rightarrow b_2 + 1$  naively comes from the presence of NS5-branes wrapped on  $\mathbf{CP}^2 \subset \mathbf{CP}^3$ , which from the  $AdS_4$  point of view look like axionic strings such that as we go around them,  $b_2 \rightarrow b_2 + 1$ . However, in the presence of  $k$  units of  $F_2$  flux, such a wrapped NS5-brane must have  $k$  D4-branes wrapped on  $\mathbf{CP}^1$  ending on it. This is consistent with the fact that inserting  $k$  fractional D4-branes is the same as taking  $b_2 \rightarrow b_2 + 1$ , since such D4-branes can annihilate to nothing by creating bubbles of wrapped NS5-branes which they can end on, and these bubbles can then shrink to nothing, leaving behind only the change  $b_2 \rightarrow b_2 + 1$ .

Furthermore, the type IIA version of the parity symmetry is given by the reflection of a coordinate in  $AdS_4$  together with  $b_2 \rightarrow -b_2$ . Combining this with  $b_2 \rightarrow b_2 + 1$  we find that the theory with  $b_2 = l/k$  should be related to the theory with  $b_2 = (k-l)/k$  by a parity transformation, implying that the  $U(N+l)_k \times U(N)_{-k}$  theory should be equivalent to the  $U(N)_k \times U(N+k-l)_{-k}$  theory. This is also in complete agreement with the field theory and M theory points of view described above.

## 2.5 A simple test

Our derivation of the M theory / string theory duals was of course far from rigorous, so we should now make various tests to see if our conjecture is sensible or not. First, note that the moduli space for a single M2-brane in our M theory background is still given by  $\mathbf{C}^4/\mathbf{Z}_k$ , in agreement with the moduli space we expect. Similarly, the spectrum of light fields in our backgrounds is not affected by the holonomy of the  $C$  field or of the  $B$  field, in agreement with our claim that the spectrum of (non-baryonic) chiral primary operators in these theories is independent of  $l$ .

What effect does this holonomy have at all, given that it does not modify the classical equations of motion? It does change the spectrum of baryonic operators. Let us analyze

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<sup>7</sup>Recall that the gauge-invariant 4-form field strength in type IIA string theory is given by  $\tilde{F}_4 = dC_3 - C_1 \wedge H_3$ .

this explicitly using the type IIA description. Recall that the background above allows for D4-branes wrapped on  $\mathbf{CP}^2 \subset \mathbf{CP}^3$ , which were identified in [5] with di-baryon operators<sup>8</sup> of the form  $(C_I)^N$ , where  $C_I$  are bi-fundamentals of  $U(N) \times U(N)$ . The worldvolume action of the D4-branes contains a coupling  $B_2 \wedge F_2 \wedge A$ , implying that when  $b_2 = l/k$ , we must have  $l$  strings ending on the D4-brane. This is exactly what we expect for the di-baryon operators in the  $U(N+l) \times U(N)$  theory, since if we take an operator like  $(C_I)^N$ , we can contract the  $U(N)$  indices to a singlet of  $SU(N)$  with an epsilon symbol, but if we then contract the indices of  $SU(N+l)$  by an epsilon symbol we are left with an object in the  $l$ 'th anti-symmetric product of fundamentals of  $SU(N+l)$ . This exactly matches the number of strings that must end on this object to form a consistent state. (Of course, the operator  $(C_I)^{N+l}$  similarly transforms in the  $l$ 'th anti-symmetric product of fundamentals of  $SU(N)$ .) Thus, the spectrum of di-baryons exactly matches what we expect.

However, we have been a bit too fast in the analysis above; what really appears in the source term on the D4-branes used above is  $F_2 \wedge (\hat{F}_2 - B_2) \wedge A$ , where  $\hat{F}_2$  is the gauge field strength on the D4-branes, and usually we say that in the presence of a  $B$  field,  $\hat{F}_2 - B_2$  is quantized rather than  $\hat{F}_2$ , so one might suspect that we must turn on some  $\hat{F}_2$  gauge field to cancel the above effect of the  $B$  field. We claim that in our background this is not correct, and we are still allowed to have configurations with  $\hat{F}_2 = 0$  that lead to the di-baryons described above. To see this, let us analyze the effect of the  $B$  field in a different way, by looking at the four dimensional effective action on  $AdS_4$ . As we discussed, the  $B$  field does not affect the equations of motion, but it does affect the action. The ten dimensional action includes a term  $B_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4$ . So, when we reduce to four dimensions, and look at the gauge field coming from  $\int_{\mathbf{CP}^1} \tilde{F}_4$ , the  $b_2$  component of the  $B$  field behaves as a theta angle for this gauge field. We know that when we shift the theta angle, the Witten effect [22] implies that the electric charges of particles are shifted by the theta angle times their magnetic charges. In the case of the specific gauge field we are discussing, the electrically charged particles are D2-branes wrapped on  $\mathbf{CP}^1$ , while the magnetically charged particles are D4-branes wrapped on  $\mathbf{CP}^2$ . So, we expect that turning on the  $B$  field will change the wrapped D2-brane charge of a wrapped D4-brane by an amount proportional to the  $B$  field. And indeed, this is exactly what happens, because of the  $B_2 \wedge C_3$  coupling on the worldvolume of the D4-branes. Note that again this coupling is really of the form  $(\hat{F}_2 - B_2) \wedge C_3$ , so that we would not get this effect if  $(\hat{F}_2 - B_2)$  were quantized; but the space-time argument above implies that we should see this effect, so we conclude that D4-brane configurations with  $\hat{F}_2 = 0$  are still allowed,<sup>9</sup> and reproduce for us the expected di-baryon spectrum discussed above. It would be interesting to reproduce this result by a direct study of the quantization of  $\hat{F}_2$  in our background.

## 2.6 Additional comments

Our conjectured duality implies that at strong coupling (large  $\lambda = N/k$ ), all the  $U(M)_k \times$

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<sup>8</sup>Note that these operators are not gauge-invariant under one of the  $U(1)$  gauge groups; nevertheless they are present in the bulk. Presumably one way to think of these operators is like electrons in QED, as defined together with a Wilson line that carries away the  $U(1)$  charge to infinity.

<sup>9</sup>This is not necessarily true for other wrapped D-branes.

$U(N)_{-k}$  theories with  $M = N, N + 1, \dots, N + k$  are very similar to each other, since they only differ by the  $B$  field, which only affects the D-brane spectrum and worldsheet instanton effects (which are suppressed by  $\exp(-\sqrt{\lambda})$ ). In particular, it seems that all these theories should be identical (at large  $N$ ) to all orders in perturbation theory in  $1/\sqrt{\lambda}$ , though not beyond this perturbation theory. For instance, the entropy of these theories should be the same to all orders in  $1/\sqrt{\lambda}$ . Of course, it is hard to check this claim, since we do not know how to perform computations in these theories at strong coupling.

Many of the computations that were performed for the  $U(N)_k \times U(N)_{-k}$  theories can easily be generalized to the  $U(M)_k \times U(N)_{-k}$  theories. In particular, it would be interesting to examine the spin chain representation of the spectrum of operators in these theories in the 't Hooft limit [9], and to see if  $(M - N)$  can be related to a  $B$  field on the “worldsheet” of the spin chain.

### 3. Orientifold theories

A class of  $\mathcal{N} = 5$  superconformal Chern-Simons-matter theories with gauge groups  $O(M) \times USp(2N)$  was recently constructed in [8]. We will review how one can obtain these theories as orientifold projections of the  $\mathcal{N} = 6$  superconformal theories, or alternatively as the IR limits of gauge theories living on D3-branes in a type IIB orientifold-brane configuration<sup>10</sup>. We will argue that these theories describe M2-branes probing the orbifold  $\mathbf{C}^4/\widehat{\mathbf{D}}_k$ , where  $\widehat{\mathbf{D}}_k$  is the binary dihedral group with  $4k$  elements, together with discrete torsion (or “fractional M2-branes”). The dual description (which is useful for  $N, M \gg k$ ) is then either M theory on  $AdS_4 \times S^7/\widehat{\mathbf{D}}_k$ , or type IIA string theory on  $AdS_4 \times \mathbf{CP}^3/\mathbf{Z}_2$ , where the  $\mathbf{Z}_2$  is an orientifold projection that we will describe below.

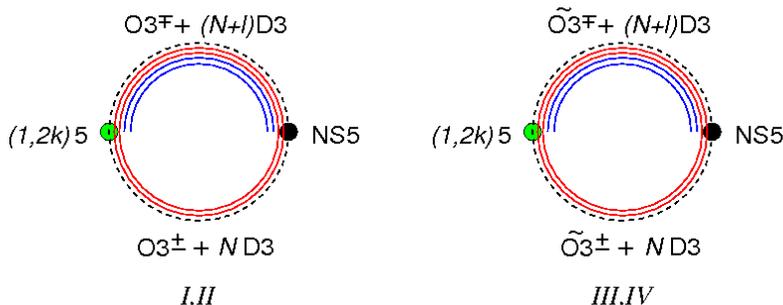
#### 3.1 Type IIB orientifold brane configurations

Let us begin with the type IIB brane configurations. The orientifold theories are obtained by including an orientifold 3-plane wrapped on the circle, in addition to the D3-branes and the two 5-branes that we had in the previous section. Adding the orientifolds does not break any additional supersymmetry. We will take the D3-branes to be “whole 3-branes” in the sense that they have images in the covering space of the orientifold (they are not identified with their own image). The 5-branes on the other hand intersect the orientifold plane, and are their own images. They are therefore “half branes” in a sense.

The orientifold 3-plane comes in four varieties, which are denoted  $O3^-$ ,  $O3^+$ ,  $\widetilde{O3}^-$  and  $\widetilde{O3}^+$ . When we have  $N$  D3-branes sitting on top of the orientifold 3-plane, these lead to the gauge groups  $O(2N)$ ,  $USp(2N)$ ,  $O(2N + 1)$  and  $USp(2N)$ , respectively. The  $O3$ -planes carry fractional D3-brane charges given by  $(-1/4)$  for  $O3^-$  and  $(+1/4)$  for  $O3^+$ ,  $\widetilde{O3}^-$  and  $\widetilde{O3}^+$ . In particular, the  $\widetilde{O3}^-$  plane can be thought of as an  $O3^-$  plane with a half D3-brane stuck on top of it. The two  $USp(2N)$  theories corresponding to  $O3^+$  and  $\widetilde{O3}^+$  are perturbatively identical, but differ in their non-perturbative (dyon) spectrum. The four

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<sup>10</sup>A different orientifold of these theories, which breaks supersymmetry, and the corresponding brane construction, were recently discussed in [23].



**Figure 3:** The brane constructions of the orientifold theories.

types of orientifolds are related by the  $SL(2, \mathbf{Z})$  duality of type IIB string theory [24]. The O3-plane variants also correspond to a choice of discrete torsion for the NSNS and RR 3-form field strengths, which take values in  $H^3(\mathbf{RP}^5, \widetilde{\mathbf{Z}}) = \mathbf{Z}_2$  [24]. In particular, NSNS torsion differentiates the  $(-)$  and  $(+)$  variants, and RR torsion differentiates the tilde and no-tilde variants. A single 5-brane that intersects an orientifold 3-plane in two spatial dimensions will therefore change it from one type on one side to a different type on the other side [25, 19]. In particular, crossing an NS5-brane changes  $O3^-$  to  $O3^+$  and  $\widetilde{O}3^-$  to  $\widetilde{O}3^+$ , and crossing a D5-brane changes  $O3^-$  to  $\widetilde{O}3^-$  and  $O3^+$  to  $\widetilde{O}3^+$ .

In our circle configuration we therefore get an orthogonal gauge group on one interval between the two 5-branes, and a symplectic gauge group on the other interval. For consistency of the configuration the D5-brane charge of the second 5-brane must be even, so we will take it to be  $(1, 2k)$ . Otherwise, the discrete torsion in the brane configuration is not single-valued. The  $O(M)$  gauge field then has a Chern-Simons term at level  $\pm 2k$ , and the  $USp(2N)$  gauge field has a Chern-Simons term at level  $\mp k$ .<sup>11</sup> We therefore find four classes of theories (see figure 3)

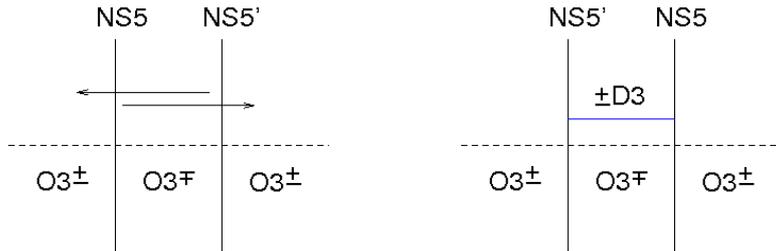
$$\begin{aligned}
 I & \quad O(2N + 2l_I)_{2k} \times USp(2N)_{-k}, \\
 II & \quad USp(2N + 2l_{II})_k \times O(2N)_{-2k}, \\
 III & \quad O(2N + 2l_{III} + 1)_{2k} \times USp(2N)_{-k}, \\
 IV & \quad USp(2N + 2l_{IV})_k \times O(2N + 1)_{-2k},
 \end{aligned} \tag{3.1}$$

and four more which are obtained by a parity transformation that takes  $k \rightarrow -k$ . We will use a shorthand notation to denote each of these theories, for example the theory in class  $I$  at level  $k$  with  $l$  fractional branes will be denoted as  $I_k(l)$ . The matter content can be determined by the action of the orientifold projection on the original  $U(2(N+l)) \times U(2N)$  bi-fundamental fields  $C_I$  ( $I = 1, 2, 3, 4$ ):

$$C_{Ia\bar{b}} \rightarrow -M_{IJ} C_{Ja\bar{c}}^* J_{c\bar{b}}, \tag{3.2}$$

where  $J$  is the invariant anti-symmetric matrix of the  $USp$  theory, and  $M_{IJ}$  is a matrix acting on the four complex scalars (and their superpartners) as  $i\sigma_2 \otimes \mathbf{1}_{2 \times 2}$ . This projects

<sup>11</sup>This gives another explanation of why we must start with  $2k$  units of D5-brane charge. The level of the  $USp(2N)$  Chern-Simons term must be an integer.



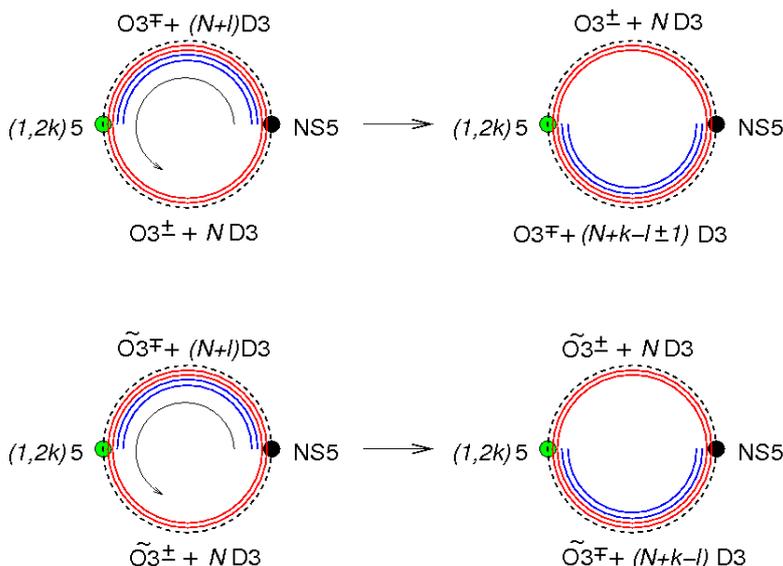
**Figure 4:** A D3-brane is created when two NS5-branes cross on an  $O3^\mp$ -plane.

out half of the two bi-fundamental hypermultiplets that we started from, leaving a single bi-fundamental hypermultiplet, or equivalently two bi-fundamental chiral superfields. For example, choosing a specific ordering of the  $C_I$ , we can take the identification on the chiral superfields  $A_i$  and  $B_i$  described in [5] to take the form  $A_1 \equiv B_1^T J$ ,  $A_2 \equiv B_2^T J$ . We therefore obtain an  $\mathcal{N} = 3$  YM-CS theory with two chiral bi-fundamental superfields  $A_1$  and  $A_2$ , and with a product of an orthogonal and symplectic gauge group in one of the classes shown in (3.1), or their parity partners. The superpotential is the projection of the original superpotential  $W \propto \text{Tr}(A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1)$  using the identifications above, which gives

$$W \propto \text{Tr}(A_1 J A_1^T A_2 J A_2^T - A_1 J A_2^T A_2 J A_1^T). \quad (3.3)$$

As in the case without the orientifold, we expect the “s-rule” and the brane creation effect to lead to a restriction on the number of inequivalent superconformal theories. However one has to be slightly careful here since both the brane creation effect, and the “s-rule” which can be derived from it, are shifted in the presence of an  $O3$ -plane relative to the case without the orientifold. The basic reason for the shift is an additional brane creation effect when two purely NS5-branes, that intersect an  $O3$ -plane, cross. In particular, if the orientifold between the NS5-branes is an  $O3^-$ -plane a single D3-brane is created, and if it is an  $O3^+$ -plane a single D3-brane is annihilated (see figure 4). If the orientifold between the NS5-branes is an  $\widetilde{O3}^-$  or  $\widetilde{O3}^+$ -plane there is no brane creation. The creation (or annihilation), or lack thereof, can be understood from the requirement that the jump in the D3-brane charge across each NS5-brane is preserved when the NS5-branes cross.<sup>12</sup> The total number of D3-branes created by the crossing NS5-brane and  $(1, 2k)5$ -brane therefore depends on the type of  $O3$ -plane between them (on the side that initially shrinks). In the  $O3^-$  case  $k + 1$  D3-branes are created, in the  $O3^+$  case  $k - 1$  are created, and in the  $\widetilde{O3}^-$  and  $\widetilde{O3}^+$  cases  $k$  D3-branes are created. This also implies that the restriction on  $l$  is shifted to  $l \leq k + 1$  in the first case (class I),  $l \leq k - 1$  in the second case (class II), and remains  $l \leq k$  in the last two cases (classes III and IV).

<sup>12</sup>One can derive this effect from a linking number analogous to the NS5-D5 case [13]. The creation of branes by NS5-branes crossing on an orientifold plane was originally proposed in type IIA brane configurations for four-dimensional  $\mathcal{N} = 1$  gauge theories with orthogonal and symplectic gauge groups in [25], in order to reproduce Seiberg duality for these theories [26]. In that case the jump in the D4-brane charge across an NS5-brane leads to an asymptotic bending of the NS5-brane, which must be preserved in finite (zero) energy brane motions.



**Figure 5:** The brane description of the parity-dualities.

We are now ready to examine the effect of moving one of the 5-branes around the circle. As before, we find that only one motion gives rise to another supersymmetric configuration, and therefore to another  $\mathcal{N} = 3$  supersymmetric gauge theory, namely the one where the 5-brane winds the circle once in the direction that initially shrinks the suspended D3-branes. The initial configuration has  $l$  suspended D3-branes in one of the segments. The final configuration has the two orientifold segments exchanged, so  $k \rightarrow -k$ , and  $k - l$  D3-branes, with the possible  $\pm 1$  shift, in the other segment. The resulting pairs of brane configurations for the four classes of theories are shown in figure 5. This leads us to conjecture the following parity-dualities between the corresponding superconformal field theories:

$$\begin{aligned}
 I_k(l) &= I_{-k}(k - l + 1), \\
 II_k(l) &= II_{-k}(k - l - 1), \\
 III_k(l) &= III_{-k}(k - l), \\
 IV_k(l) &= IV_{-k}(k - l).
 \end{aligned}
 \tag{3.4}$$

This identification is consistent with the bounds described above:

$$0 \leq l_I \leq k + 1, \quad 0 \leq l_{II} \leq k - 1, \quad 0 \leq l_{III,IV} \leq k.
 \tag{3.5}$$

Looking at (3.1) we also see that at the extremal values the parity-dualities of (3.4) give equivalence relations that map between the different classes:

$$\begin{aligned}
 I_k(k + 1) &= II_k(0), \\
 II_k(k - 1) &= I_k(0), \\
 III_k(k) &= IV_k(0), \\
 IV_k(k) &= III_k(0).
 \end{aligned}
 \tag{3.6}$$

All in all, we therefore have  $4k$  inequivalent theories (for each value of  $N$  and  $k$ ).

As in the case with the unitary gauge groups, the two theories related by a parity-duality are never weakly coupled at the same time, making it difficult to test this duality. However we will provide more evidence for these dualities, as well as for the maximal values of  $l$ , by analyzing the superconformal theories directly, and we will also show that this picture is consistent with the gravitational dual descriptions.

### 3.2 The $O(M) \times \text{USp}(2N)$ superconformal theories

The three-dimensional orthogonal/symplectic  $\mathcal{N} = 3$  gauge theories living on the brane configurations described above flow in the IR to superconformal Chern-Simons-matter theories. Alternatively we can also get these superconformal theories by gauging a discrete symmetry of the  $\mathcal{N} = 6$  superconformal theories with  $U(\tilde{M})_{2k} \times U(\tilde{N})_{-2k}$  which includes the usual orientifold action on the gauge fields, and the action on the matter fields in (3.2). This action breaks the baryon number symmetry  $U(1)_b$  completely, while the subgroup of the R-symmetry  $SU(4)_R$  left unbroken by  $M_{IJ}$  is  $\text{USp}(4) \sim \text{SO}(5)$ . The new  $O(M) \times \text{USp}(2N)$  gauge theory must therefore have  $\mathcal{N} = 5$  supersymmetry, and this was verified (classically) in [8].<sup>13</sup>

Let us now determine the moduli space of these theories. A complete analysis of the moduli space of these theories is given in the appendix. Here we will motivate the result by a shortcut, starting with the moduli space of the parent  $\mathcal{N} = 6$  superconformal theory. The latter is given by  $(\mathbf{C}^4/\mathbf{Z}_{2k})^{\tilde{N}}/\mathbf{S}_{\tilde{N}}$ , where the  $\mathbf{Z}_{2k}$  is generated by the transformation  $C_I \rightarrow e^{\pi i/k} C_I$ . It is easy to see that the symmetry group of (3.2) is  $\mathbf{Z}_4$ . This symmetry does not commute with the  $\mathbf{Z}_{2k}$  symmetry acting on the moduli space of the original theory. In fact, the square of the  $\mathbf{Z}_4$  generator acts trivially on the gauge fields, and on the matter fields it gives  $C_I \rightarrow -C_I$ , which corresponds precisely to the generator of the  $\mathbf{Z}_{2k}$  symmetry raised to the  $k$ 'th power. This is precisely the definition of the binary dihedral group  $\widehat{\mathbf{D}}_k$  (we are using the convention that  $\widehat{\mathbf{D}}_k$  is the binary dihedral group with  $4k$  elements). The simplest one is  $\widehat{\mathbf{D}}_1 = \mathbf{Z}_4$ , and the higher  $k$  groups are non-Abelian. Thus, we expect the moduli space after the projection to be  $(\mathbf{C}^4/\widehat{\mathbf{D}}_k)^N/\mathbf{S}_N$ , and we verify in the appendix that this is indeed the case.

Based on this moduli space, it is natural to conjecture that our theories are related to the theory of  $N$  M2-branes probing the orbifold  $\mathbf{C}^4/\widehat{\mathbf{D}}_k$  in M theory. This orbifold preserves precisely  $\mathcal{N} = 5$  supersymmetry [28], which is consistent with this conjecture. This conjecture implies that the dual M theory background is  $AdS_4 \times S^7/\widehat{\mathbf{D}}_k$ . We also expect a regime of the parameters  $N$  and  $k$  in which there is a weakly curved type IIA string theory description. We will discuss both gravitational duals below.

But first, let us return to the implications of the “s-rule” and brane creation effect, namely the parity-duality relations (3.4), and the maximal values of  $l$  (3.5). We will use

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<sup>13</sup>Note that one could also consider  $\mathcal{N} = 5$  supersymmetric theories with the gauge group  $\text{SO}(M) \times \text{USp}(2N)$ , but we do not obtain these theories from our brane construction or from the projection of the unitary theories, and we will not consider them in this paper. For the special case of  $\text{SO}(2) \times \text{USp}(2N)$  these theories have an enhanced  $\mathcal{N} = 6$  supersymmetry [8, 27].

similar arguments to the ones we used for the  $U(N+l) \times U(N)$  theories as further evidence for these quantum properties of the orientifold theories.

We start by considering a generic point on the moduli space. This leaves at low energies a pure  $\mathcal{N} = 3$  supersymmetric Chern-Simons theory with a gauge group that depends on  $l$ . We can then integrate out the gauginos to get an effective bosonic Chern-Simons theory with a shifted level, where the shift is given by the dual Coxeter number of the gauge group,  $k \rightarrow k - h \cdot \text{sign}(k)$  [14].<sup>14</sup> The dual Coxeter numbers of the groups  $O(2l)$ ,  $O(2l+1)$  and  $USp(2l)$  are given, respectively, by  $h = 2l - 2$ ,  $2l - 1$  and  $l + 1$ . We can then determine the maximal value of  $l$ , as in the previous section, by requiring that the level in the effective bosonic CS theory should have the same sign as the original level, so that the one-loop shift of the level in this theory will bring us back to the original level  $k$ . This gives  $0 \leq l_I \leq k+1$ ,  $0 \leq l_{II} \leq k - 1$  and  $0 \leq l_{III} \leq k$ , for the class I, II and III theories, respectively, precisely as predicted above.

The class IV theories are a bit more subtle, because on their moduli space we obtain a  $USp(2l_{IV}) \times O(1)$  gauge theory; the  $O(1)$  theory does not have any gauge fields, but it leads to massless matter fields in the fundamental of  $USp(2l_{IV})$ . Thus, we cannot map these theories directly to bosonic CS theories. We believe that these superconformal theories exist precisely for  $0 \leq l_{IV} \leq k$ , and it would be interesting to verify this directly.

Our main direct evidence for the parity-dualities in (3.4) comes (as in the unitary case) from level-rank duality between the two effective bosonic Chern-Simons theories on the moduli space. For example, the  $I_k(l)$  theory gives at a generic point on the moduli space an  $\mathcal{N} = 3$   $O(2l)_{2k}$  Chern-Simons theory, which is equivalent to a bosonic  $O(2l)_{2k-2l+2}$  Chern-Simons theory, while the  $I_{-k}(k-l+1)$  theory on the moduli space includes an  $\mathcal{N} = 3$   $O(2k-2l+2)_{-2k}$  theory, which is equivalent to a bosonic  $O(2k-2l+2)_{-2l}$  theory. The two WZW theories that we obtain from these in the presence of boundaries are related by level-rank duality, which supports the first of the four conjectured dualities in (3.4). A similar test of the  $II_k(l)$  and  $III_k(l)$  dualities confirms their consistency as well. For the class IV theories it is again more difficult to analyze this directly because of the presence of extra massless fields, so we do not have direct field theory arguments for the duality in this case.

### 3.3 M theory lift

The configuration of D3-branes, O3-planes, and fivebranes described above can be T-dualized and lifted to M theory as in [5], resulting in M2-branes probing a singularity in a toric hyperKähler 8-manifold, which preserves 3/16 of the supersymmetry [29]. This geometry has a  $T^2$  formed from the T-dualized circle and the M theory circle, fibred over a six dimensional base. The IR limit of the field theory on the D3-branes that we derived corresponds to taking the size of the wrapped circle small in IIB, and hence large in the T-dual picture. Further, the low energy limit of the D2-branes brings us to strong coupling, so the superconformal field theory describes the limit of large  $T^2$  in the hyperKähler geometry.

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<sup>14</sup>In [14] the shift was by  $-\frac{h}{2} \cdot \text{sign}(k)$  for the  $\mathcal{N} = 1$  theory with one gaugino. In the  $\mathcal{N} = 3$  theory there are four gauginos, three of which contribute with the same sign, and the fourth with the opposite sign.

In [5], the hyperKähler geometry resulting from the theory with one NS5-brane and one  $(1, k)$  5-brane was described explicitly, and its IR limit was shown to be a  $\mathbf{C}^4/\mathbf{Z}_k$  singularity. The hyperKähler geometries were studied in [29], and have a general form related to a product of two Taub-NUT spaces,

$$\begin{aligned} ds^2 &= U_{ij} d\vec{x}^i \cdot d\vec{x}^j + U^{ij} (d\varphi_i + A_i)(d\varphi_j + A_j), \\ A_i &= d\vec{x}^j \cdot \vec{\omega}_{ji} = dx_a^j \omega_{ji}^a, \\ \partial_{x_a^j} \omega_{ki}^b - \partial_{x_b^k} \omega_{ji}^a &= \epsilon^{abc} \partial_{x_c^j} U_{ki}, \end{aligned} \tag{3.7}$$

where  $U_{ij}$  is a two by two symmetric matrix of harmonic functions and  $U^{ij}$  is its inverse. In our case there will be an additional  $\mathbf{Z}_2$  quotient due to the lift of the orientifold.

Suppose we first consider the T-dual and lift of the O3-plane without any 5-branes. Then, in M theory the geometry would simply be a  $\mathbf{Z}_2$  quotient,<sup>15</sup> acting on the coordinates above as  $\vec{x}_i \rightarrow -\vec{x}_i$ , and  $\varphi_i \rightarrow -\varphi_i$ , where  $\vec{x}_i$  describe a flat  $\mathbf{R}^6$ , and  $\varphi_i$  describes the fiber  $T^2$ . Re-introducing the fivebranes, the torus fiber will shrink along 3-planes in the base. However, far from these centers of the Taub-NUT spaces, the geometry is not altered significantly, and one sees that the  $\mathbf{Z}_2$  action must be the same as above.

The particular geometry obtained from an NS5-brane and a  $(1, 2k)$  5-brane is described by

$$U = \mathbf{1} + \begin{pmatrix} h_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} h_2 & 2kh_2 \\ 2kh_2 & 4k^2h_2 \end{pmatrix}, \quad h_2 = \frac{1}{2|\vec{x}_1 + 2k\vec{x}_2|}, \quad h_1 = \frac{1}{2|\vec{x}_1|}, \tag{3.8}$$

in which the singularity at the origin is  $\mathbf{C}^4/\mathbf{Z}_{2k}$ , as explained in [5]. Note that near the origin of the hyperKähler 8-manifold, one obtains  $\mathbf{C}^4/\mathbf{Z}_{2k}$  using the fact that the center of each Taub-NUT geometry looks just like a flat  $\mathbf{R}^4$  space, naturally written in polar coordinates with a Hopf-fibred angular  $S^3$ .

More explicitly, the Hopf fibration defines a map  $f : \mathbf{C}^2 \rightarrow S^2$  which can be written as  $f(z_1, z_2) = (2\text{Re}(z_1 z_2^*), 2\text{Im}(z_1 z_2^*), |z_1|^2 - |z_2|^2)$ , where the  $S^2$  is regarded as the unit sphere in  $\mathbf{R}^3$ , the base of the Taub-NUT. The orientifold  $\mathbf{Z}_2$  then acts as the antipodal map on the  $S^2$ , which can be seen to lift to the action  $z_1 \rightarrow iz_2^*$ ,  $z_2 \rightarrow -iz_1^*$  on  $\mathbf{C}^2$ , noting that the overall phase of  $z_1, z_2$  is exactly the Hopf circle, which is also inverted,  $\varphi_1 \rightarrow -\varphi_1$ , by the lift of the orientifold. This  $\mathbf{Z}_2$  also acts simultaneously on the other  $\mathbf{C}^2$  in the eight dimensional geometry.

Therefore we conclude that the transverse geometry to the M2-branes includes a  $\mathbf{C}^4/\widehat{\mathbf{D}}_k$  singularity (which will control the IR field theory). The  $\mathbf{Z}_{2k}$  subgroup of  $\widehat{\mathbf{D}}_k$  is the orbifold obtained from the 5-branes, acting as  $(z_1, z_2, z_3, z_4) \rightarrow e^{\frac{\pi i}{k}}(z_1, z_2, z_3, z_4)$ , while the other generator acts by

$$(z_1, z_2, z_3, z_4) \rightarrow (iz_2^*, -iz_1^*, iz_4^*, -iz_3^*). \tag{3.9}$$

The latter is a  $\mathbf{Z}_2$  transformation on the  $\mathbf{C}^4/\mathbf{Z}_{2k}$  orbifold, but of course it gives a  $\mathbf{Z}_4$  in the  $\mathbf{C}^4$  covering space.

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<sup>15</sup>More precisely, we obtain four such quotients on four different points on the  $T^2$ , but we will focus here on the vicinity of one of these four orbifold points.

It is not hard to check that this is identical to the  $\mathbf{C}^4/\widehat{\mathbf{D}}_k$  singularity mentioned in [28], in different coordinates. One sees that the  $\widehat{\mathbf{D}}_k$  sits inside an  $SU(2)$  subgroup of the  $SO(8)$  rotation group of  $\mathbf{R}^8$ , and thus its commutant is the  $SO(5)_R$  R-symmetry of this  $\mathcal{N} = 5$  orbifold.

In the special case of  $k = 1$ , the orbifold is simply  $\mathbf{C}^4/\mathbf{Z}_4$ , which is unique up to coordinate redefinitions. Therefore the  $O \times USp$  theories actually preserve  $\mathcal{N} = 6$  supersymmetry for  $k = 1$ , and they should be quantum mechanically dual to the  $\mathcal{N} = 6$  Chern-Simons-matter theories with unitary gauge groups (with the same minimal rank) at level  $k = 4$ . In particular, the four  $O \times USp$  theories in this case are  $O(2N)_2 \times USp(2N)_{-1}$ ,  $O(2N+2)_2 \times USp(2N)_{-1}$ ,  $O(2N+1)_2 \times USp(2N)_{-1}$ , and  $USp(2N)_1 \times O(2N+1)_{-2}$ . The first two should then presumably be equivalent to the  $U(N)_4 \times U(N)_{-4}$  and  $U(N+2)_4 \times U(N)_{-4}$  theories (though we cannot say which to which), and the last two to  $U(N+1)_4 \times U(N)_{-4}$  and  $U(N+3)_4 \times U(N)_{-4}$ . Of course, these theories are strongly coupled, so it is difficult to check these identifications, but one can compare their spectrum of chiral operators, their moduli spaces, and so on. As a simple consistency check, note that the parity-dualities (3.4), (3.6) imply that the first two  $O \times USp$  theories mentioned above map to themselves by a parity transformation. This is obviously true also for the  $U(N)_4 \times U(N)_{-4}$  theory, and it turns out to also be true for the  $U(N+2)_4 \times U(N)_{-4}$  theory, using the equivalences (2.1). Similarly, our last two  $O \times USp$  theories map to each other under parity, and using (2.1) we can check that the same is true for the  $U(N+1)_4 \times U(N)_{-4}$  and  $U(N+3)_4 \times U(N)_{-4}$  theories.

### 3.4 M theory dual

From the M theory lift above, it is easy to construct the M theory background dual to our orientifolded field theories (3.1). This background is simply  $AdS_4 \times S^7/\widehat{\mathbf{D}}_k$ , with  $N$  units of 4-form flux in  $AdS_4$ :

$$\begin{aligned}
 ds^2 &= \frac{R^2}{4} ds_{AdS_4}^2 + R^2 ds_{S^7/\widehat{\mathbf{D}}_k}^2, \\
 G_4 &\sim N\epsilon_4,
 \end{aligned}
 \tag{3.10}$$

where  $\epsilon_4$  is the unit volume form on  $AdS_4$ . This is obviously consistent with the moduli space of these field theories which we described above. The spectrum of chiral operators in this background was recently computed in [30]. The eleven dimensional supergravity approximation is valid for  $k \ll N^{1/5}$ , as before.

It is natural to describe the  $S^7/\widehat{\mathbf{D}}_k$  as a  $\mathbf{Z}_2$  quotient of the  $S^1$  fibred over  $\mathbf{CP}^3$  that we had in the  $S^7/\mathbf{Z}_{2k}$  case. This is equivalent to a twisted circle bundle over the non-orientable 6-manifold  $\mathbf{CP}^3/\mathbf{Z}_2$  (where by twisted we mean that the curvature of the circle bundle lives in the  $\mathbf{Z}_2$ -twisted cohomology).

The geometry described above has a  $\mathbf{Z}_{4k}$  torsion 3-cycle; generally, the discrete torsion in M theory when we divide by a discrete group with  $|\Gamma|$  elements is  $\mathbf{Z}_{|\Gamma|}$  [31]. One way to describe this cycle explicitly is as the twisted  $S^1$  fibration over an  $\mathbf{RP}^2$  in the base. This cycle is precisely half the torsion cycle in  $S^7/\mathbf{Z}_{2k}$ , given by the  $S^1$  fibred over a  $\mathbf{CP}^1$ . In M theory, one can thus turn on a discrete torsion  $G_4$  flux, or in other words, a flat  $C_3$  field

with holonomy in  $\mathbf{Z}_{4k} \subset \text{U}(1)$  on that torsion 3-cycle. Explicitly,

$$C_3 \propto \frac{l}{4k} J \wedge d\varphi, \tag{3.11}$$

in the  $S^7/\widehat{\mathbf{D}}_k$ , with  $l = 0, \dots, 4k - 1$ ; this is invariant under the identifications described above. Thus, we have  $4k$  different field theories, just as expected from our discussion above. Upon reduction to type IIA string theory, we will see precisely how this discrete torsion matches with the  $4k$  orientifold field theory duals found earlier.

### 3.5 Type IIA string theory orientifold dual

When  $N \ll k^5$  the M theory circle becomes small, and it is more appropriate to describe the gravitational dual using type IIA string theory (which is weakly curved for  $N \gg k$ ). The reduction to type IIA is very similar to the one discussed in the previous section, (2.5) (with  $2k$  replacing  $k$  everywhere). The only difference is the additional  $\mathbf{Z}_2$  identification which we have to take into account. This identification acts on the natural projective coordinates of  $\mathbf{CP}^3$  by (3.9). It is easy to check that this maps  $J \rightarrow -J$ , so that the orientation of  $\mathbf{CP}^3$  (and, thus, of the full type IIA background) is reversed. Thus, our type IIA background is an orientifold of  $AdS_4 \times \mathbf{CP}^3$  (the orientifold has no fixed points so this is a smooth manifold). In M theory, the  $\mathbf{Z}_2$  acts on the  $\mathbf{CP}^3$  as above, together with an inversion of the coordinate on the M theory circle (while not acting on the 3-form field). Thus, in type IIA string theory, beyond its geometric action described above, the orientifold inverts the RR 1-form  $C_1$  and the NSNS 2-form  $B_2$ , while leaving invariant the RR 3-form  $C_3$ . Note that since  $F_2 \rightarrow -F_2$  and  $J \rightarrow -J$ , this is consistent with the RR 2-form flux  $F_2 = 2kJ$  (which is invariant under the  $\mathbf{Z}_2$ ). One can think of this as  $k$  units of RR 2-form flux on the cycle  $\mathbf{CP}^1/\mathbf{Z}_2$  that we obtain by performing the identification (3.9) on the  $\mathbf{CP}^1 \subset \mathbf{CP}^3$ . Note that this identification maps the  $\mathbf{CP}^1$  (defined, say, by  $z_3 = z_4 = 0$ ) to itself, so that we obtain a smaller 2-cycle, but it does not map the  $\mathbf{CP}^2$  cycle (defined, say, by  $z_4 = 0$ ) to itself. Thus, the minimal 4-cycle in the orientifolded background is still  $\mathbf{CP}^2$ .

Both  $\text{U}(1)$  gauge fields which are present on  $AdS_4 \times \mathbf{CP}^3$  before the orientifold, are projected out by the orientifold (one of them originally became massive by swallowing an axion [5], but this axion is also projected out by the orientifold). The projection on the  $\text{SU}(4)$  gauge fields leaves an  $\text{SO}(5)$  subgroup invariant, so that the gauge group on  $AdS_4$  agrees with the global symmetry group of our field theories.

Next, let us analyze the reduction of discrete torsion in this picture. The fluxes  $f_4$  and  $b_2$  described in the previous section are invariant under the  $\mathbf{Z}_2$ . As in the previous section we can define  $f_4 \propto \int_{\mathbf{CP}^2} \tilde{F}_4$  (normalized to be an integer in the absence of 2-form flux), but we can now define a smaller 2-form holonomy  $\tilde{b}_2 = \frac{1}{2\pi} \int_{\mathbf{CP}^1/\mathbf{Z}_2} B_2$  (which is periodic with period one). The same arguments as in the previous section imply that the quantized four-form flux is  $f_4 + 4k\tilde{b}_2$  (since we have  $2k$  units of  $F_2$  flux on  $\mathbf{CP}^1$ , and  $\tilde{b}_2 = \frac{1}{2}b_2$ ). Thus, we have  $4k$  possibilities for the NSNS holonomy (when  $\tilde{F}_4 = 0$ ), given by  $\tilde{b}_2 = l/4k$  with  $l = 0, \dots, 4k - 1$  (up to a possible integer shift). This precisely agrees with our M theory expressions in the previous subsection.

As in the previous section, the fact that the different  $I_k(l)$  theories are related by adding fractional branes (each of which shifts  $\tilde{b}_2$  by  $\frac{1}{2k}$ ) makes it natural to identify the  $I_k(l)$  theory with the type IIA background with  $\tilde{b}_2 = \frac{l}{2k}$ . However, the fractional brane picture only tells us the differences between the values of  $\tilde{b}_2$  between the different theories, and not their absolute values, so it allows for an identification of the form  $\tilde{b}_2 = \frac{l+c}{2k}$  with some (half)-integer  $c$ . The  $III_k(l)$  theory is then related to this by adding another “half of a fractional brane” (in the brane construction), so it is natural to identify it with the background with  $\tilde{b}_2 = \frac{2l+2c+1}{4k}$ . Using the equivalence relations (3.6), we can then determine the identifications of the class II and class IV theories. Our full identifications are thus summarized by:

$$\begin{aligned}
 I_k(l) &\leftrightarrow \tilde{b}_2 = \frac{l+c}{2k}, \\
 II_k(l) &\leftrightarrow \tilde{b}_2 = \frac{l+c+k+1}{2k}, \\
 III_k(l) &\leftrightarrow \tilde{b}_2 = \frac{l+c+\frac{1}{2}}{2k}, \\
 IV_k(l) &\leftrightarrow \tilde{b}_2 = \frac{l+c+k+\frac{1}{2}}{2k}.
 \end{aligned}
 \tag{3.12}$$

For any value of  $c$  we have a one-to-one map between our space of theories and the space of type IIA backgrounds. In order to determine  $c$ , we require consistency with the parity-duality identifications of (3.4). Recall that the parity transformation of type IIA takes  $\tilde{b}_2 \rightarrow -\tilde{b}_2 = 1 - \tilde{b}_2$ , and we conjectured that it acts on our theory space as (3.4). All of these identifications turn out to be consistent for precisely two values of  $c$ :  $c = -\frac{k+1}{2}$  or  $c = \frac{k-1}{2} \pmod{2k}$ . Note that the NSNS holonomy encodes in the language of our original type IIB brane construction both the number of fractional branes and the NSNS and RR torsion fluxes, as a result of the chain of dualities that we performed.<sup>16</sup>

It is not clear how to determine which of the two possibilities for  $c$  is correct. Note that for both possibilities, the  $I_k(0)$  theory does not map to  $\tilde{b}_2 = 0$  as one might have naively expected; presumably the orientifold itself induces some discrete torsion in our background, even in the absence of any fractional branes. This is consistent with the fact that the  $I_k(0)$  theory is not parity-invariant (for  $k > 1$ ). It would be interesting to understand this better, and to determine the precise value of  $c$ . Since we have no massless  $U(1)$  fields, the di-baryon operators (coming from D4-branes wrapped on  $\mathbf{CP}^2$ ) are not independent of the “meson”-type operators, in agreement with our gauge theory expectations. Thus, these operators do not seem to give any clues about the correct identifications.

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<sup>16</sup>The type IIA orientifold may also admit RR torsion corresponding to a discrete holonomy of the RR 3-form  $C_3$  on a torsion 3-cycle in  $\mathbf{CP}^3/\mathbf{Z}_2$  ( $H_3(\mathbf{CP}^3/\mathbf{Z}_2, \mathbf{Z}) = \mathbf{Z}_2$ ). Since we have no room for an extra quantum number associated with this, we expect that this torsion should be correlated with the parity of  $4k\tilde{b}_2$ , that distinguishes the class I and II theories from the class III and IV theories (and is related to the RR torsion in our brane construction). This correlation may be seen in M theory by noting that the torsion 3-cycle lifts to the same torsion cycle in M theory as  $\mathbf{CP}^1/\mathbf{Z}_2$ . It would be interesting to derive this correlation directly in type IIA string theory by a careful analysis of the flux quantization conditions in our background.

It would be interesting to analyze the mass deformations of these theories (analogous to [12]) and to try to obtain a correct count of the number of vacua in these theories. It would also be interesting to see if analyzing the operators in this theory in the 't Hooft large  $N$  limit leads to interesting integrable spin chains, either on the field theory or on the string theory side. Of course, at the leading (planar) order the theories of this section are identical to those of the previous section. The leading difference between them comes from non-orientable diagrams at the next order in  $1/N$ .

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## A. Moduli spaces of the orientifold field theories

In this appendix we compute the classical moduli spaces of the  $O(2M)_{2k} \times \text{USp}(2N)_{-k}$  and  $O(2M+1)_{2k} \times \text{USp}(2N)_{-k}$  field theories discussed above; these are expected to be exact quantum mechanically due to the considerable amount of supersymmetry. Recall that the moduli space of the  $U(N) \times U(M)$  Chern-Simons-matter theory consists of diagonal matrices [5], so one might expect the same here. However, the orientifold projection (3.2) does not preserve the diagonal form due to the presence of  $J$ . This is reflected in the fact that it is impossible to use the projected  $O \times \text{USp}$  gauge symmetry to diagonalize a generic matrix in the bi-fundamental, in contrast to the unitary group case.

We will choose  $J$  to be made of  $2 \times 2$  blocks in which it is equal to  $(-i\sigma_2)$ . Using the gauge symmetry, it is then possible to bring the vacuum expectation value of one of the complex bi-fundamental matter fields,  $A_1$ , into a  $2 \times 2$  block diagonal form. It is easy to check that on the moduli space  $A_2$  must also have this form. Within each block we then just need to determine the moduli space of the  $O(2) \times \text{USp}(2)$  theory. In the first part of this appendix we show that the moduli space of this theory is  $\mathbf{C}^4/\widehat{\mathbf{D}}_k$ , on which the gauge group is generically broken to a  $U(1)$ , in agreement with our interpretation of this theory

as describing a single M2-brane in that geometry; moreover, that is the result expected by projecting the moduli space of the  $U(2N) \times U(2N)$  theory found in [5].

Therefore, the moduli space of the  $O(M)_{2k} \times \text{USp}(2N)_{-k}$  theories for  $M \geq 2N$  is given by  $N$  copies of the  $O(2) \times \text{USp}(2)$  moduli space, quotiented by the unbroken permutation symmetry  $S_N \subset O(2N) \times \text{USp}(2N)$ , namely it is equal to  $Sym^N(\mathbf{C}^4/\widehat{\mathbf{D}}_k)$ . The low-energy effective theory on this moduli space includes a residual pure  $\mathcal{N} = 3$  Chern-Simons theory  $O(M - 2N)_{2k}$ .

For the  $\text{USp}(2M)_k \times O(2N)_{-2k}$  theories with  $M \geq N$  the moduli space is similarly given by  $Sym^N(\mathbf{C}^4/\widehat{\mathbf{D}}_k)$ , where now the low-energy effective action on the moduli space includes an  $\mathcal{N} = 3$  supersymmetric  $\text{USp}(2M - 2N)_k$  pure Chern-Simons theory. The situation for the  $\text{USp}(2M)_k \times O(2N + 1)_{-2k}$  theories is similar, except that the low-energy effective theory on the moduli space now includes an  $\mathcal{N} = 3$  supersymmetric Chern-Simons-matter theory with gauge group  $\text{USp}(2M - 2N)_k \times O(1)_{-2k}$ , which has massless matter in addition to the  $\mathcal{N} = 3$  Chern-Simons terms. In the second part of this appendix we show that there are no additional moduli associated to this residual theory, so it does not affect the moduli space.

### A.1 The moduli space of the $O(2)_{2k} \times \text{USp}(2)_{-k}$ theory

We begin by gauge fixing the gauge fields to zero, so that the moduli space is given by the zero locus of the bosonic potential for the two bi-fundamental scalar fields  $A_1, A_2$  (which we write as  $2 \times 2$  matrices). This potential can be written as a sum of squares, coming both from the usual F-terms  $|\partial W|^2$  (with  $W$  given by (3.3)), and from the  $\sigma$ -terms in the supersymmetric kinetic terms. These terms are proportional to

$$\sum_{a=1}^2 \text{Tr}(|\sigma_O A_a - A_a \sigma_{\text{USp}}|^2), \tag{A.1}$$

where the scalar fields  $\sigma$  in the vector multiplet are equal (up to a constant that we will ignore) to the moment maps,

$$\sigma_O = \sum_{a=1}^2 (A_a A_a^\dagger - A_a^* A_a^T), \quad \sigma_{\text{USp}} = \sum_{a=1}^2 (A_a^\dagger A_a + J A_a^T A_a^* J), \tag{A.2}$$

after integrating out the auxiliary fields in the vector multiplets. Their form can either be derived directly, or by using the projection from the unitary case.

It is easy to see that both the F-term equations

$$A_1^T A_2 J A_2^T = A_2^T A_2 J A_1^T, \quad A_2^T A_1 J A_1^T = A_1^T A_1 J A_2^T, \tag{A.3}$$

and the equations coming from (A.1) are satisfied when  $[J, A_1] = [J, A_2] = 0$ , since in that case the matrices take the form

$$A_1 = x + yJ, \quad A_2 = z + wJ, \tag{A.4}$$

so that the matrices (and their conjugates and transposes) all commute with each other. The scalar fields in the vector multiplets are then equal to

$$\sigma_{O(2)} = 2(yx^* - xy^* + wz^* - zw^*)J = \sigma_{\text{USp}(2)}, \tag{A.5}$$

so it is easy to check that (A.1) vanishes. Therefore, the matrices (A.4) are on the moduli space for any complex numbers  $x, y, z, w$ .

Generically, on the moduli space, the gauge symmetry is broken to a diagonal  $U(1)$ , acting by an  $O(2)$  transformation on the left, and by the inverse transformation (which is contained in  $\text{USp}(2)$ ) on the right. The unbroken gauge field is thus  $\mathcal{A}_+ = \mathcal{A}_{O(2)} + \mathcal{A}_2$ , where we define  $\mathcal{A}_{\text{USp}(2)} = \sum_{j=1}^3 \sigma_j \mathcal{A}_j$ . The other three generators of the gauge group become massive. However, the off-diagonal combination  $\mathcal{A}_- = \mathcal{A}_{O(2)} - \mathcal{A}_2$  acts on the moduli space, that is, it preserves the form (A.4). This rotation acts on the components as

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{A.6}$$

and similarly for  $z$  and  $w$ . It is convenient to define new coordinates on the moduli space  $u_1 = \text{Re}(x) + i\text{Re}(y)$ ,  $u_2 = \text{Im}(x) + i\text{Im}(y)$ ,  $u_3 = \text{Re}(z) + i\text{Re}(w)$ ,  $u_4 = \text{Im}(z) + i\text{Im}(w)$ , so that this broken  $U(1)$  acts by multiplying all the  $u_I$  ( $I = 1, 2, 3, 4$ ) by  $e^{i\phi}$ .

Next, we need to take into account the effect of constant gauge transformations (which leave the gauge fields vanishing) acting on the moduli space. Since effectively we have a  $U(1) \times U(1)$  theory, this analysis is precisely the same as in [5]. The only difference is that the level of the  $U(1)$  Chern-Simons terms is now  $2k$  instead of  $k$  as in [5]; this is clear for the  $O(2)_{2k}$  piece, and one can show that it is also true for the  $U(1)$  in  $\text{USp}(2)_{-k}$  by considering the embedding of  $U(1)$  in  $\text{USp}(2)$ . Thus, by a similar analysis to [5] (either by considering which gauge transformations leave the CS terms invariant, or by dualizing the gauge field  $\mathcal{A}_+$  to a scalar), we find that the  $U(1)$  action leads to a  $\mathbf{Z}_{2k}$  identification, where the  $\mathbf{Z}_{2k}$  acts as  $u_I \rightarrow \exp(\pi i n/k) u_I$  (for  $n \in \mathbf{Z}$ ).

This takes into account the connected part of the gauge group. However,  $O(2)$  has an additional disconnected component (involving matrices with determinant  $(-1)$ ), and this leads to an additional  $\mathbf{Z}_2$  identification of the moduli space. In particular, we can consider a gauge transformation

$$A_1 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -iy & -ix \\ ix & -iy \end{pmatrix}, \tag{A.7}$$

(and similarly for  $A_2$ ). This preserves the form (A.4), and acts on the coordinates defined above as  $(u_1, u_2, u_3, u_4) \rightarrow (iu_2^*, -iu_1^*, iu_4^*, -iu_3^*)$ . This is exactly the orientifold action we discussed in section 3, which squares to  $u_I \rightarrow -u_I$  which is an element of the  $\mathbf{Z}_{2k}$  that we already identified. Thus, after imposing all the identifications, the moduli space is precisely  $\mathbf{C}^4/\widehat{\mathbf{D}}_k$ .

## A.2 The moduli space of the residual $\text{USp}(2l)_k \times O(1)_{-2k}$ theories

In these theories we can think of the massless matter fields simply as fundamentals of  $\text{USp}(2l)$ , since the  $O(1)$  gauge group is just a discrete gauged  $\mathbf{Z}_2$ . It is enough to analyze the case of  $l = 1$ , since the analysis for general  $l$  will just be  $l$  copies of this.

The F-term equations imply that  $A_1^T A_2 J A_2^T = A_2^T A_2 J A_1^T$ , but since  $A_2$  is just a vector we have that  $A_2 J A_2^T = 0$  for any  $A_2$ . Thus  $A_2 J A_1^T = 0$  (or  $A_2 = 0$ ), and similarly  $A_1 J A_2^T = 0$  (or  $A_1 = 0$ ). We conclude that  $A_1$  and  $A_2$  must be proportional to each other,  $A_2 = \alpha A_1$  for some  $\alpha \in \mathbf{C}$ .

It is obvious that  $\sigma_{O(1)} = 0$ , since there is no  $O(1)$  Lie algebra, while  $\sigma_{\text{USp}}$  is equal to  $\sigma_{\text{USp}} = A_1^\dagger A_1 + J A_1^T A_1^* J + A_2^\dagger A_2 + J A_2^T A_2^* J$ . The constraint from the  $\sigma$ -term of the first field is then

$$0 = A_1 \sigma_{\text{USp}} = A_1 A_1^\dagger A_1 + A_1 A_2^\dagger A_2 = (1 + |\alpha|^2)(A_1 A_1^\dagger) A_1, \quad (\text{A.8})$$

since  $A_2$  is proportional to  $A_1$ , and  $A_1 J A_1^T = 0$ . This is impossible unless  $A_1 = A_2 = 0$ , since  $(1 + |\alpha|^2)(A_1 A_1^\dagger)$  is positive. Thus, the only classical solution to all the constraints is  $A_1 = A_2 = 0$ .

We conclude that there is no moduli space associated to this residual theory, in spite of the presence of massless fundamentals at its unique classical vacuum.

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